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Analytical calculation of pile lateral deformations and maximum stresses: Application to structural modelling

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ABSTRACT

This work revisits the methodology to calculate the exact equivalent pile length to be used in the soil-structure interaction modeling during the design process. The formulation is presented using the background of analytical equations to calculate the equivalent pile length and an analytical equation to calculate the maximum longitudinal bending moment in the pile. This formulation admits a link between the elastic center and the equivalent rigid pile length, either for a singular or group of piles. This methodology is compared with existing ones proposed by other authors that also provide a comparable method when calculating the lateral deformations of piles. It also tested some analytical expressions normally used to estimate the maximum longitudinal bending moment and compared them with the proposed one in this work. For validating the proposed model, it is compared with known benchmarks and full 3D solid finite element models, in which the advantages of the proposed methodology are highlighted.

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Equivalent pile length; Soil-Structure interaction; exact stiffness; stresses in piles; group Pile-Soil system

1. Introduction

During the pre-design phase, the evolution of stress in the group pile-soil system is essential to estimate a robust and economic structural solution. Although several methodologies already exist to estimate the global stiffness of pile-soil infrastructure, most of these employ several simplifications, which later need to be corrected during the final design phase. The lateral pile-head response of a horizontal loaded pile is typically approximated using one of two types of comparable models: the uncoupled spring model or the equivalent cantilever model. Because the flexibility matrix of these similar models does not exactly match that of the original Winkler pile-soil model, they cannot function correctly for various pile configurations and loading scenarios. The response will never provide accurate forces and displacements at the same time.

In terms of flexibility, most of these existing methods tend to either obtain an unclear upper or lower bound value when compared with the analytical values of a model for a pile with elastic springs in deepness. This is even clearer when estimating the maximum bending moment in the pile [1], since according to the authors' best knowledge, there is no clear proposal, and finite element models are always necessary to enquire its values. This problem tends to be aggravated and, in some extreme cases, may even present an artificial dynamic response due to adopted wrong stiffness [2].

Although it is always possible to use a full finite element model to correctly simulate the soil-pile interaction either by Winkler model with elastic springs or by using full 3D solid

models, these may not be cost effective especially during the pre-design phase, in which approximations are in order.

Although it is known the difficulty in testing "*in situ*" full scale lateral load of piles [3], in recent years the scientific community has gained some interest in the evaluation of the lateral stiffness of soil-pile systems.

Although it is possible to use 3D finite element commercial software [4], some of which were especially developed for soil and foundation engineering [5] to design piles, engineers still need methods to validate and confirm the results, without needing to believe blindly in the computer outputs. Indeed, this type of analysis may even be more complicated for non-conventional loads in piles [6]. These validating methodologies include the equivalent pier and equivalent raft methods, which, as [7] and [8] have shown, can both offer sufficient solutions for pile group settlement when used correctly.

This topic has been brought up since new research found that seasonally frozen soil changes its lateral reaction in the long term, which may affect the dynamic behavior of bridges [9] and consequently its effective design life period due to traffic vibrations. New methodologies for estimating the horizontal stiffness, based on standards regulating the assignment of the proportionality factors, were proposed [10]. These needed some adaptation due to the inelastic behavior of the soil, that may compromise the reaction or stiffness of the equivalent linearly deformable soil bed. This problem is even extended to the vertical settlement of an isolated pile, in which some authors have recommended a simple design procedure in the form of the equivalent pier

approach [11], by adjusting numerical results of observed “*in situ*” experimental outputs. A variation of this research was fulfilled [12,13], in this case for group piles either for mechanical or thermal loads, using an extensive parametric finite element campaign.

1.1. Objectives

The main purpose of this work is to provide an alternative method to calculate an exact equivalent pile length that can calculate either the exact flexibility of the pile-soil system or an estimate of the maximum longitudinal bending moment in the pile.

The main advantage of this methodology is that it can be used either for the pre-design phase using a simple linear elastic model. Or even in more advanced non-linear analysis using 3D solid models with interface surfaces between the pile and soil, by modifying the equivalent stiffness values by taking into account the general non-linear response of the pile-soil system either by the constitutive relation of the pile or/and soil.

1.2. Research significance

Although methodologies already exist to correctly estimate the exact equivalent model [14], these are mainly valid to estimate the foundation flexibility and do not allow the estimation of the maximum bending moment in the longitudinal deepness of the pile. This methodology allows both the estimation of the foundation flexibility and the calculation of the maximum bending moment applied to the pile. This is performed by admitting a link between the elastic center and the equivalent pile length.

With this proposed methodology, structural designers will have a simple tool to be used during the design phase and compared with more advanced models, without any of the simplifications usually found in these types of models. This work plans to fill this gap in knowledge.

2. Existing pile-soil models

In this section it is presented the previous pile in elastic medium formulation already known by the scientific community. It is also presented several models that analyze the assessment of the flexibility of a pile-soil system. It is beyond the scope of this work to present all existing popular methods, and these are usually divided in either subgrade reaction approach using the Winkler formulation or full continuum approach. Both can be solved with classical finite element tools, and with some modifications, they can also estimate the vertical resistance. For further detail in this topic it is recommend to consult the recent work presented by Moussa et al. [15].

2.1. Pile in elastic medium

The theoretical foundations of beam behavior in elastic medium are not novel, and it has been fully studied and

detailed in Hetenyi[16]. Assuming physically and geometrically linear behavior for a beam in elastic soil Figure 1, in which this last one can be admitted as a constant continuum elastic spring ($k_s = cte$), its structural response can be derived from the following ordinary differential Eq. (1). It is beyond this work, but a full demonstration of this procedure can be found in Edwards et al. [17], in which the complex domain $\mathbb{C}$ must be used to derive the roots and solution of the differential Eq. (2).

$$EI \frac{d^2 w(x)}{dx^2} + k_s w(x) = 0 \quad (1)$$

$$w(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + C_3 e^{r_3 x} + C_4 e^{r_4 x} \quad (2)$$

with

$$\lambda = \sqrt[4]{\frac{k_s}{4EI}}$$

$$r_1 = -r_3 = \lambda(i + 1)$$

$$r_2 = -r_4 = \lambda(i - 1)$$

The calculation of soil reaction k_s is not direct and may depend on its nature due to possible variation in deepness. The easiest way to estimate it is to use the expression adopted by Vesic [18], for a beam in an elastic foundation (Eq. 3) with diameter D_p with infinite length. According to Castro [19], for circular concrete piles, this leads to values ranging from $0.4E_s < k_s < 0.6E_s$.

$$k_s = 0.65 \sqrt[12]{\frac{E_{soil} D_p^4}{E_c I}} \cdot \frac{E_{soil}}{D_p (1 - \nu^2)} \quad (3)$$

The variable λ includes the bending flexibility of the pile as well as the stiffness of the soil. Because of this, the variable λ is referred to as the characteristic of the system, and since its dimension is L^{-1} , the inverse term (Eq. 4) is called the characteristic length of the pile, L_c .

$$L_c = \frac{1}{\lambda} \quad (4)$$

The solution of this differential Eq. (2) can be simplified (Eq. 5) for a pile with infinite length ($\lambda L \geq \pi$) inserted in soil, and admitting only applied loads in the top head [20].

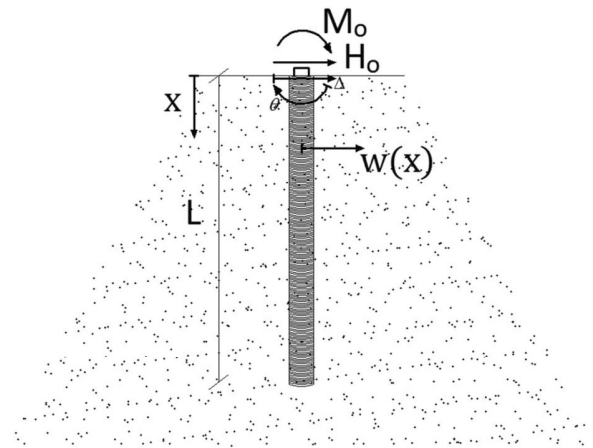


Figure 1. Schematic of applied load, adopted axis and displacement in the head of the pile.

Assuming linear global behavior, it can be use the principle of linear superposition to separate the effects of horizontal shear and bending moment in the pile (Eq. 6). Therefore, the structural performance of each load separately can be analyzed (Eq. 7), using the same positive general axis representation (Eq. 8).

$$w(x) = e^{\lambda x}(C_1 \cos \lambda x + C_2 \sin \lambda x) + e^{-\lambda x}(C_3 \cos \lambda x + C_4 \sin \lambda x) \quad (5)$$

$$\begin{cases} w(x) = w_H(x) + w_M(x) \\ \theta(x) = \theta_H(x) + \theta_M(x) \\ M(x) = M_H(x) + M_M(x) \\ V(x) = V_H(x) + V_M(x) \end{cases} \quad (6)$$

$$\begin{cases} w_H(x) = \frac{2H_0\lambda}{k_s} D(x) \\ \theta_H(x) = \frac{2H_0\lambda^2}{k_s} A(x) \\ M_H(x) = \frac{H_0}{\lambda} B(x) \\ V_H(x) = H_0 C(x) \end{cases} \begin{cases} w_M(x) = \frac{2M_0\lambda^2}{k_s} C(x) \\ \theta_M(x) = \frac{4M_0\lambda^3}{k_s} D(x) \\ M_M(x) = M_0 A(x) \\ V_M(x) = 2M_0\lambda B(x) \end{cases} \quad (7)$$

$$\begin{aligned} A(x) &= e^{-\lambda x}(\cos \lambda x + \sin \lambda x) & B(x) &= e^{-\lambda x}(\sin \lambda x) \\ C(x) &= e^{-\lambda x}(\cos \lambda x - \sin \lambda x) & D(x) &= e^{-\lambda x}(\cos \lambda x) \end{aligned} \quad (8)$$

To get the maximum bending moment, it is necessary to analyze the first derivative (Eq. 9). It is easy to separate the structural response for each applied load to better understand its contribution (Eq. 10). The final expression to find the position x_m (Eq. 11), cannot be solved using analytical methods. The simplest way to find the position x_m is for each load separately.

$$M_{\max}(x_m) \Rightarrow \frac{dM(x)}{dx} \Big|_{x_m} = 0 \quad (9)$$

$$\frac{dM_{H_0}(x)}{dx} \Big|_{x_m} + \frac{dM_{M_0}(x)}{dx} \Big|_{x_m} = 0 \quad (10)$$

$$(H_0 + \lambda M_0)(\cos \lambda x_m - \sin \lambda x_m) = \lambda M_0(\cos \lambda x_m + \sin \lambda x_m) \quad (11)$$

First, let's consider two different separate cases ($H_0; M_0 = 0$) and ($H_0 = 0; M_0$), than the previous Eq. (11)

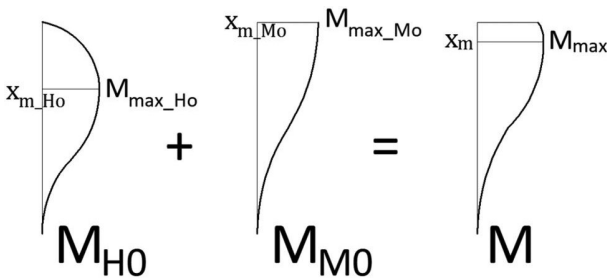


Figure 2. Qualitative bending moment diagram for the horizontal load and applied bending moment at the top head of the pile.

presents an analytical solution (Eq. 12). Using these results, with the general deformation solution presented in Figure 2 [21], and admitting that both loads are increasing the internal stresses in the same direction, the position of x_m can be bound by the given interval (Eq. 13).

$$\begin{aligned} \frac{dM_{H_0}(x)}{dx} \Big|_{x_m} &= 0 \Rightarrow x_m = \frac{\pi}{4\lambda} \\ \frac{dM_{M_0}(x)}{dx} \Big|_{x_m} &= 0 \Rightarrow x_m = 0 \end{aligned} \quad (12)$$

$$0 \leq x_m \leq \frac{\pi}{4\lambda} \quad (13)$$

For the case only with the horizontal load applied in the pile's head ($H_0; M_0 = 0$), due to the soil iteration with the pile, it can be an overestimation to admit the maximum bending moment as (14).

$$M_{\max}^{H_0} \ll H_0 \frac{\pi}{4\lambda} \simeq 0.7854 \frac{H_0}{\lambda} \quad (14)$$

For each separate load, it is possible to find out the position at which the maximum longitudinal bending moment is found (Eq. 15).

$$\begin{aligned} M_{\max}^{H_0}(x_{mH_0} = \pi/4\lambda) &= \frac{H_0}{\sqrt{2}\lambda} e^{(-\pi/4)} \simeq 0.3224 \frac{H_0}{\lambda} \\ M_{\max}^{M_0}(x_{mM_0} = 0) &= M_0 \end{aligned} \quad (15)$$

$$M_{\max}(x_m) \leq 0.3224 \frac{H_0}{\lambda} + M_0 \quad (16)$$

It is therefore possible to admit an upper bound (Eq. 16) to estimate the maximum longitudinal bending moment in an isolated pile, although this one being a non-economic design choice if used by the structural engineer.

One initial estimate for the maximum longitudinal bending moment in the pile as proposed by Cintra [20]. This expression admits that the horizontal load presents higher stress influence in the pile (Eq. 17), which is normal the case for group piles. However, if the horizontal load contribution is lower than the applied bending moment in the pile's head, than this proposal may underestimate its contribution. This is the case in elevated pile caps or isolated direct to column to pile systems.

$$M_{\max}^{approx}(x_m) \simeq 0.3224 \frac{H_0}{\lambda} + 0.7M_0 \quad (17)$$

2.2. Equivalent pile model

To generate an equivalent pile model, it is necessary to first understand how the equivalence works in terms of flexibility and what physical outputs can indeed be used. Admitting a pile inserted in an elastic foundation, an equivalent pile length is searched, admitting it is fixed in a fictitious rigid soil (encastrated) Figure 3 using the same pile section.

To assemble the equivalent pile length, the stiffness of the degrees of freedom (Eq. 18) of the pile-soil system must be

equal to the ones adopted for a cantilever with a free head in the top of Figure 4.

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \begin{bmatrix} K_{\Delta\Delta} & K_{\Delta\theta} \\ K_{\theta\Delta} & K_{\theta\theta} \end{bmatrix} \quad (18)$$

The values of the stiffnesses for the cantilever are known [22] and for linear analysis are described in expression (19). It is important to point out, that there are at least three independent stiffnesses for the pile-soil system, meaning the equivalent pile length needs at least three independent variables during its definition, otherwise the adopted stiffness values are not exact [23].

$$K_{\Delta\Delta} = \frac{12EI}{L_{eq}^3}; K_{\Delta\theta} = K_{\theta\Delta} = -\frac{6EI}{L_{eq}^2}; K_{\theta\theta} = \frac{4EI}{L_{eq}} \quad (19)$$

The stiffness and flexibility values for the pile-soil system are also known [16,21] and can be derived from expression (5) for the horizontal displacement and rotation at the top of the pile's head (Eq. 24).

$$K_{ij} = \begin{bmatrix} 4EI\lambda^3 & -2EI\lambda^2 \\ -2EI\lambda^2 & 2EI\lambda \end{bmatrix} F_{ij} = \begin{bmatrix} 0.5(EI\lambda^3)^{-1} & 0.5(EI\lambda^2)^{-1} \\ 0.5(EI\lambda^2)^{-1} & (EI\lambda)^{-1} \end{bmatrix} \quad (20)$$

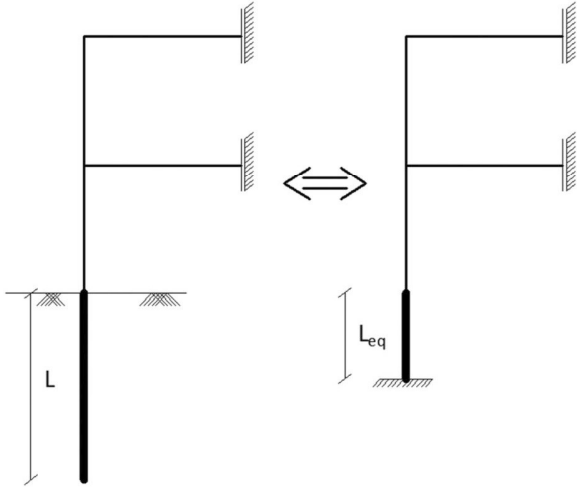


Figure 3. Equivalent pile length as a cantilever for the structural modeling.

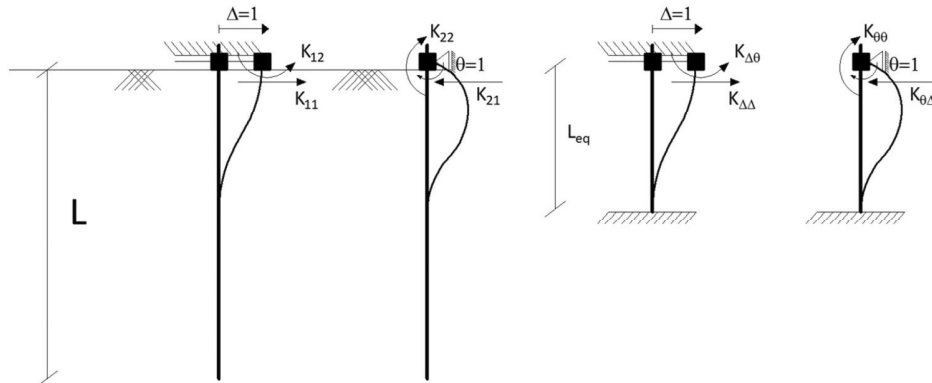


Figure 4. Independent generalized displacements for the pile-soil system and the equivalent pile.

In which:

$$\begin{aligned} Q_1 &= H; Q_2 = M \\ q_1 &= \Delta; q_2 = \theta \\ F_{ij}Q_j &= q_i \Rightarrow K_{ij}q_j = Q_i \end{aligned} \quad (21)$$

By equalizing the terms of the stiffness matrices Eq. (20) and Eq. (19) and admitting a simple equivalent column with the given equivalent pile length, it is possible to verify that this is not a unit equal value (Eq. 22). According to this analogy, the equivalent pile length can vary from $1.45L_c$ to $2.00L_c$.

$$\begin{aligned} K_{11} = K_{\Delta\Delta} &\Rightarrow \frac{4EI}{L_c^3} = \frac{12EI}{L_{eq}^3} \Rightarrow L_{eq} = 1.45L_c \\ K_{12} = K_{\Delta\theta} &\Rightarrow \frac{2EI}{L_c^2} = \frac{6EI}{L_{eq}^2} \Rightarrow L_{eq} = 1.73L_c \\ K_{22} = K_{\theta\theta} &\Rightarrow \frac{2EI}{L_c} = \frac{4EI}{L_{eq}} \Rightarrow L_{eq} = 2.00L_c \end{aligned} \quad (22)$$

According to some initial studies from Davisson et al. [25], it should adopt a singular value of $1.00L_c$, in which it is possible to obtain a similar value of maximum bending moment of a pile embedded in elastic springs. However, this last assumption admits that both loads are applied separately, and only an upper bound is used to take into account the joint effect of pile bending and buckling, making a non-economic structural design.

Some authors [26] for pre-design purposes declare that some simplifications can be adopted, and an elastic center for an unknown type of soil can be approximated by Eq. (23), if the diameter D_p is known in a group pile. According to this author, this y_c can be used as an elastic length to estimate the maximum bending moment in a pile-soil system with fixed rotation [27]. This simplification is only valid if $k_s \simeq E_{soil}$, therefore ignoring the general expression in Eq. (3), for soils with constant elasticity in depth and if the pile is circular. Also, this simplification provides no guidance on what flexibility is to be used for the equivalent pile length and is only recommended to be applied to estimate an upper bound of the bending moment in the pile. It is also possible to numerically calculate the maximum bending moment in the pile using classical Fourier integral solutions [28], but as stated, this method tends to overestimate the

real value. This simplification can also be extrapolated using the same assumptions for soils with increasing linear elasticity with deepness (Eq. 24).

$$L_{eq} \simeq y_c \simeq \frac{D_p}{3} \sqrt[4]{\frac{E_{concrete}}{E_{soil}}} \quad (23)$$

$$L_{eq} \simeq y_{ch} \simeq 0.471 D_p^{4/5} \sqrt[5]{\frac{E_{concrete}}{E_{soil}}} \quad (24)$$

$$M_{\max}(x) \approx H_0 y_c + M_0 \quad (25)$$

For the case of a soil with linear elastic variation in depth ($k_s = n_h x$), using the same formulation with the flexibility of the pile-soil system, it is possible to calculate equivalent pile length. The characteristic of the system is now named λ_h (Eq. 26), but the flexibility (Eq. 27) cannot be calculated analytically, although a numerical approximation can be found in Poulos et al. [21].

$$\lambda_h = \sqrt[5]{\frac{n_h}{EI}} = \frac{1}{L_{ch}} \quad (26)$$

$$\begin{bmatrix} H \\ M \end{bmatrix} = \begin{bmatrix} 1.076EI\lambda_h^3 & -0.997EI\lambda_h^2 \\ -0.997EI\lambda_h^2 & 1.497EI\lambda_h \end{bmatrix} \begin{bmatrix} \Delta \\ \theta \end{bmatrix} \quad (27)$$

To numerically calculate the maximum longitudinal bending moment in the pile, it is necessary to use the non-dimensional parameters present in the abacos of the report published by Matlock et al. [29], using the expression (28).

$$M(x) = C_{H_0}(x\lambda_h)L_{ch}H_0 + C_{M_0}(x\lambda_h)M_0 \quad (28)$$

Agreeing to this analogy, the equivalent pile length can vary from $2.24L_c$ to $2.67L_c$, these are illustrated in expression (29). Again, according to Davisson and Nair [24,25], for this particular case it should be adopted a singular value of $1.8L_c$.

$$\begin{aligned} K_{11h} = K_{\Delta\Delta} &\Rightarrow \frac{1.076EI}{L_{ch}^3} = \frac{12EI}{L_{eq}^3} \Rightarrow L_{eq} = 2.24L_{ch} \\ K_{12h} = K_{\Delta\theta} &\Rightarrow \frac{0.997EI}{L_{ch}^2} = \frac{6EI}{L_{eq}^2} \Rightarrow L_{eq} = 2.45L_{ch} \\ K_{22h} = K_{\theta\theta} &\Rightarrow \frac{1.497EI}{L_{ch}} = \frac{4EI}{L_{eq}} \Rightarrow L_{eq} = 2.67L_{ch} \end{aligned} \quad (29)$$

2.3. Previous existing models

It is beyond the scope of this work to cite all the existing models, to estimate the equivalent pile length, or to equivalent spring model. As stated before, different criteria will yield different equivalent pile lengths.

These different formulations are unable to attain a unique equivalency. The loading pattern that is transferred from the superstructure to the pile itself will determine whether either equivalency is applicable. For engineering analysis, which typically involves a variety of loading combinations, it is not practical for design purposes. These even depend on if the

elasticity of the soil is either constant ($k_s = cte$) or it varies linearly with deepness ($k_s = n_h x$).

The easiest way an uncoupled spring model [30] Figure 5(b). Due to this simplification, this model ignores the coupling effects between the lateral displacement and rotation at the pile head and instead simulates the pile-head stiffness of a lateral pile-soil system using a set of uncoupled lateral and rotational springs. The coefficients of rotational spring K_θ and transitional spring K_Δ are obtained directly from the diagonal terms of the single pile-head stiffness matrix of a lateral pile-soil system in engineering practice.

As presented before, a cantilever with an equivalent length L_{eq} Figure 5(a) was proposed by several authors [24] as a way to model the pile head flexibility of a lateral single pile-soil system (Eq. 22). Only some terms of the pile-head flexibility can be fitted by the simulation because only one parameter is employed for equivalency. Variations of this equivalent length L_{eq} were also proposed [31,32] by yielding different criteria.

Another study conducted by PoLam et al. [33] proposed using a cantilever with an equivalent flexural rigidity EI_{eq} and an equivalent length L_{eq} to approximate pile-head stiffness in a lateral single pile-soil system Figure 5(c). Obviously, despite the fact that two parameters were used in their formulation, they are insufficient to fit all three (two diagonal and one cross-coupling) coefficients in the relevant stiffness matrix at once. Two approaches are possible, either by matching the two diagonal terms of the pile-head stiffness matrix or by matching the translational and the cross-coupling terms.

Outstanding work was performed by Chiou et al. [14], in which for the first time it was possible to calculate an exact equivalent pile length model by adopting three different independent variables Figure 5(d). The variables the author chose were L_{eq} , EI_{eq} and a lateral translational stiffness in the bottom of the pile. However, this model was not able to obtain an expression of the maximum bending moment in the pile since the original EI was modified by EI_{eq} . The best procedure to estimate the maximum bending moment in the pile with this method is to use the expression (30).

$$M_{\max} = H_0 L_{eq} + M_0 \quad (30)$$

A brief summary of these models is displayed in Tables 1 and 2, but for more detailed information regarding their assumptions, simplifications and background formulation, the original work should be consulted.

3. Proposed formulation

The proposed methodology to analyze the flexibility of a pile-soil system and the estimation of the maximum bending moment in a pile is based on the elastic center of a rigid frame with displacement and rotational springs Figure 6.

Three independent variables are defined as $K_\Delta; K_\theta; y_c$, since the pile-soil system always presents three unknown stiffnesses $K_{\Delta\Delta}; K_{\Delta\theta} = K_{\theta\Delta}; K_{\theta\theta}$ (Eq. 19). The variable y_c is associated with the total length of the rigid frame and is an unknown that will influence the global flexibility of the

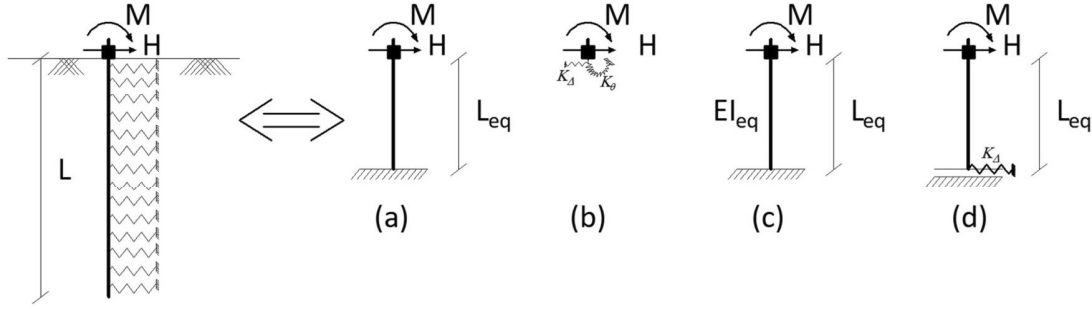


Figure 5. Types of equivalent pile-soil system flexibilities.

Table 1. For soils with constant elasticity ($k_s = cte$).

Name	Expressions
Uncoupled Spring (US) [30]	$K_\Delta = 4EI\lambda^3; K_\theta = 2EI\lambda$
Eq. Cantilever (LG) [26]	$L_{eq} = \frac{D_p}{3} \sqrt{\frac{E_{concrete}}{E_{soil}}}$
Eq. Cantilever (LUEQ) [23]	$L_{eq} = 1.45L_c$
Eq. Cantilever (LREQ) [23]	$L_{eq} = 2.00L_c$
Eq. Cantilever (LC) [25]	$L_{eq} = L_c$
Eq. Cantilever (KUEQ) [33]	$L_{eq} = 1.732K_{22}^{0.5}K_{11}^{-0.5}; EI_{eq} = 0.433K_{22}^{1.5}K_{11}^{-0.5}$
Eq. Cantilever (KDEQ) [33]	$L_{eq} = -2.0K_{12}K_{11}^{-1}; EI_{eq} = -0.667K_{12}^3K_{11}^{-2}$
Exact Eq. Model (EEM) [14]	$L_{eq} = 1/\lambda; EI_{eq} = EI; K_\Delta = 6EI\lambda^3$

Table 2. For soils with elasticity that varies linear with depth. ($k_s = n_h x$).

Name	Expressions
Uncoupled Spring (UCh) [30]	$K_\Delta = 1.077EI\lambda_h^3; K_\theta = 1.485EI\lambda_h$
Eq. Cantilever (LGh)	$L_{eq} = 0.471D_p^{4/5} \sqrt{\frac{E_{concrete}}{E_{soil}}}$
Eq. Cantilever (LUEQh) [23]	$L_{eq} = 2.24L_{ch}$
Eq. Cantilever (LREQh) [23]	$L_{eq} = 2.67L_{ch}$
Eq. Cantilever (LCh) [25]	$L_{eq} = 1.8L_{ch}$
Exact Eq. Model (EEMh) [14]	$L_{eq} = 1.839/\lambda_h; EI_{eq} = 1.057EI; K_\Delta = 2.28EI\lambda_h^3$

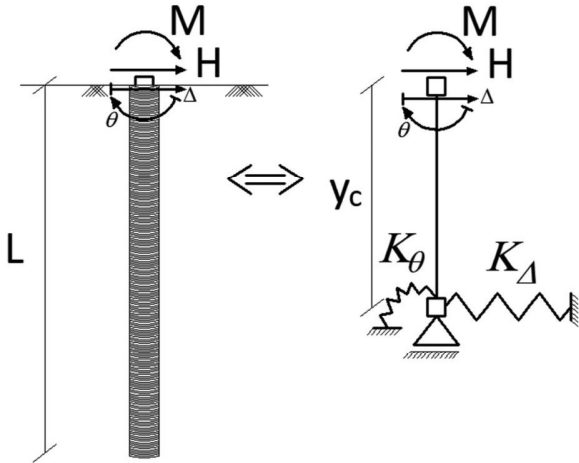


Figure 6. Proposed equivalent pile length model.

equivalent model. This variable is commonly known as the elastic center [34]. The easiest way to deduce the connection between the stiffnesses is to first calculate the flexibility matrix for each applied load, H_0 and M_0 separately (Eq. 31).

3.1. Equivalent pile length

The simplest way to calculate the flexibility matrix of the equivalent pile length model is to apply the principle of

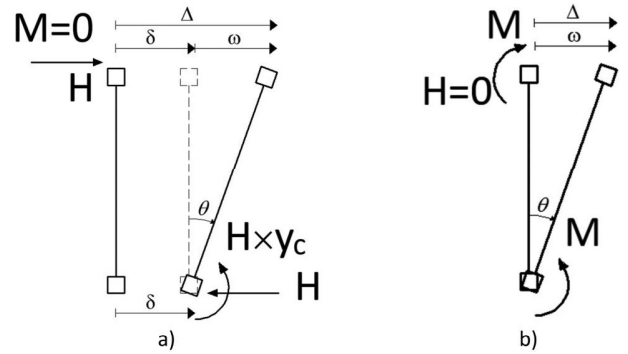


Figure 7. Flexibility of the proposed equivalent pile length model.

virtual displacements (PVD). First applying $H = \bar{1}; M = 0$ in each horizontal displacement Δ is a joint effect of the translation δ and translation ω due to rotation θ Figure 7(a). Next, it is applied as $H = 0; M = \bar{1}$, but in this case the displacement Δ is only affected due to translation ω from rotation θ Figure 7(b).

$$\begin{aligned} H_1 = 1; M_1 = 0 &\Rightarrow F_{11} = \delta_1 + \omega_1 = \Delta_1, F_{12} = \theta_1 \\ H_2 = 0; M_2 = 1 &\Rightarrow F_{21} = \omega_2 = \Delta_2, F_{22} = \theta_2 \end{aligned} \quad (31)$$

It is possible to know for linear analysis the value of the bending moment in the rotational spring (Eq. 32), and therefore calculate the value of the flexibilities of the equivalent pile model (Eq. 33). These are generalized for unitary values as Eq. (31), and each one must be equal to the flexibility of the pile-soil system (Eq. 34) previously displayed in Eq. (20).

$$\theta = \frac{H_1 y_c}{K_\theta} \text{ or/and } \frac{M_{K_\theta}}{K_\theta} \quad (32)$$

$$\begin{aligned} F_{11} &= \frac{H_1}{K_\Delta} + \theta y_c; F_{12} = \frac{H_1 y_c}{K_\theta} \\ F_{21} &= \frac{M_1 y_c}{K_\theta}; F_{22} = \frac{M_1}{K_\theta} \end{aligned} \quad (33)$$

$$F_{ij} = \begin{bmatrix} \frac{1}{K_\Delta} + \frac{y_c^2}{K_\theta} & \frac{y_c}{K_\theta} \\ \frac{y_c}{K_\theta} & \frac{1}{K_\theta} \end{bmatrix} = \begin{bmatrix} \frac{1}{2EI\lambda^3} & \frac{1}{2EI\lambda^2} \\ \frac{1}{2EI\lambda^2} & \frac{1}{EI\lambda} \end{bmatrix} \quad (34)$$

Due to the symmetry of the problem (Eq. 35), the equivalence from flexibility to stiffness is direct and can be deduced using Betti's theorem Timoshenko et al. [35], to

connect to the stiffness of the equivalent pile model (Eq. 36).

$$\det(F_{ij}) = F_{11}F_{22} - F_{12}^2; F_{12} = F_{21} \quad (35)$$

$$\begin{cases} K_\theta = (F_{22})^{-1} \\ y_c = \frac{F_{12}}{F_{22}} \\ K_\Delta = \frac{F_{22}}{\det(F_{ij})} \end{cases} \iff \begin{cases} K_\theta = \frac{\det(K_{ij})}{K_{11}} \\ y_c = -\frac{K_{12}}{K_{11}} \\ K_\Delta = K_{11} \end{cases} \quad (36)$$

For a soil with constant elasticity in depth ($k_s = cte$), the values of the stiffness can be analytically calculated (Eq. 37), and these are clearly not in line with the previous estimation in section 2. In the case for a general pile, independently of the applied load or pile head restrictions, there is a clear connection between the elastic center y_c and the pile characteristic length L_c .

$$\begin{cases} K_\theta = EI\lambda = \frac{EI}{L_c} \\ y_c = \frac{1}{2\lambda} = \frac{L_c}{2} \\ K_\Delta = 4EI\lambda^3 = \frac{4EI}{L_c^3} \end{cases} \quad (37)$$

The elastic center will be onwards called the equivalent pile length of the proposed equivalent pile model to simplify the presentation of the work when comparing pile models from other authors. The expressions for the stiffness of the equivalent pile in Eq. (38) are still valid for a soil with linear elastic variation in depth ($k_s = n_h x$) and should be adopted to calculate the translational and rotational stiffness.

$$\begin{cases} K_\theta = 0.573EI\lambda_h = \frac{0.573EI}{L_{ch}} \\ y_c = \frac{0.927}{\lambda_h} = 0.927L_{ch} \\ K_\Delta = 1.076EI\lambda_h^3 = \frac{1.076EI}{L_{ch}^3} \end{cases} \quad (38)$$

3.2. Maximum pile bending moment

Using the expression (16), just like stated previously in Eq. (14), the equivalent pile length y_c can represent a smaller upper bound of the maximum bending moment in the pile (Eq. 39), when only a horizontal load H_0 is applied. This upper bound once again is due to the soil iteration with the pile, in which an overestimation can be admitted for the maximum bending moment as Eq. (40).

$$M_{\max}^{H_0} = 0.64478y_c H_0 < y_c H_0 < 1.5708y_c H_0 \quad (39)$$

$$M_{\max}(x_m) \leq 0.64478y_c H_0 + M_0 \quad (40)$$

It can now be stated that the previous expression (25) is indeed an upper bound for the bending moment. It is still possible to analyze the position of the maximum bending moment at the pile if the limits are bonded to small values

of H_0 and M_0 . Therefore, the position of the maximum bending moment is in the interval (Eq. 43).

$$\begin{cases} \lim_{M_0 \rightarrow 0} M_{\max}(x) = 0.64478H_0y_c \\ \lim_{M_0 \rightarrow 0} x_m = 1.5708y_c \\ \lim_{H_0 \rightarrow 0} M_{\max}(x) = M_0 \\ \lim_{H_0 \rightarrow 0} x_m = 0 \end{cases} \quad (41)$$

$$\begin{cases} \lim_{H_0y_c \ll M_0} M_{\max}(x) = M_0 \\ \lim_{H_0y_c \ll M_0} x_m = 0 \end{cases} \quad (42)$$

$$0 \leq x_m \leq 1.5708y_c \quad (43)$$

With this information, it is possible to propose one expression to estimate the maximum bending moment in a pile (Eq. 44). This expression guarantees that the limits (Eqs. 41, 42) and (Eq. 43) are verified.

$$M_{\max}(x_m) \simeq 0.64478H_0y_c \frac{H_0}{H_0 + \lambda M_0} + M_0 \quad (44)$$

For the case of a soil with linear elastic variation in depth ($k_s = n_h x$), it is harder to estimate the maximum bending moment since no analytical expression exists. But, by observing the pile stresses and deformation variation in the abacos published in [21,36] and represented in Figure 2, it is still possible to propose a similar proportional expression (45), although it lacks scientific demonstration as the previous one. The correction with the square root is due to the maximum bending moment in the pile being located deeper when compared with soil with constant elasticity.

$$M_{\max}(x_m) \approx 0.833H_0y_c \sqrt{\frac{H_0}{H_0 + \lambda_h M_0}} + M_0 \quad (45)$$

3.3. Application to group piles

The adopted methodology is still possible to apply for a general group pile system with soil with different properties and layers in depth Figure 8. The flexibility matrix (Eq. 34) can be calculated using the PVD, and if the deformations are controlled, the non-linear behavior can be implemented in the equivalent springs Eq. (46). The procedure is the same as the one presented in section 3.1 applying first $H = \bar{1}; M = 0$ in each horizontal displacement Δ is a joint effect of the translation δ and translation ω due to rotation θ . Afterwards it is applied $H = 0; M = \bar{1}$, but in this case the displacement Δ is only affected due to translation ω from rotation θ .

$$\begin{aligned} H = 1; M = 0 &\implies F_{11} = \theta_1; F_{12} = \Delta_1 \\ H = 0; M = 1 &\implies F_{21} = \theta_2; F_{22} = \Delta_2 \end{aligned} \quad (46)$$

This can be performed directly on either a 2D or 3D finite element model, with either frame or solid elements to model the pile-soil system. If 3D solid elements are used for either the pile and soil, with the correct interface for the interaction between pile and soil, even the group effect appears indirectly in the equivalent springs Eq. (36).

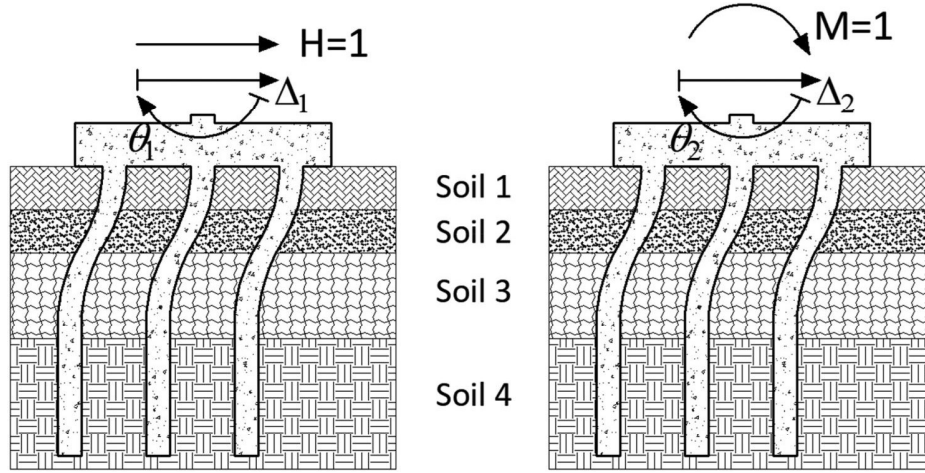


Figure 8. Calculation of the flexibility of the group pile with different soil interaction.

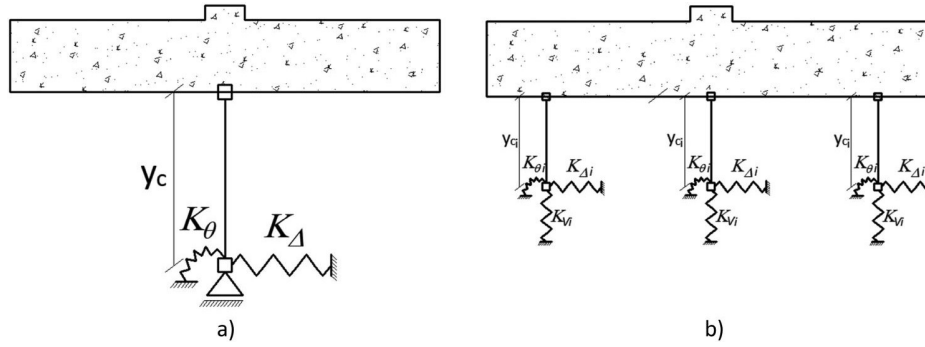


Figure 9. Models for equivalent pile length: (a) a global model for stiffness; (b) a localized model.

For group piles there are two possibilities. The first is using directly a global group flexibility Figure 9 a), but this modeling choice does not allow to calculate the bending moment in each isolated pile. The second approach is to calculate the flexibility of each isolated pile Figure 9(b), and for each isolated pile an equivalent length is adopted Eq. (37). In the first approach only transitional and rotational springs are needed. However, for the second approach, it is not kinematically possible to estimate the group rotational flexibility without vertical springs since the equivalent pile length is rigid. The vertical stiffness isn't referred to in this work K_v , these can be calculated for group piles according to the findings in Norris [37].

4. Numerical examples

In this section, some numerical examples will be presented to test the proposed formulation and methods provided by other authors. Initially it will only be tested as an isolated pile, subjected to both horizontal load and bending moment in the top head. The models will be compared in terms of deformation and maximum stresses found in the pile for the different approaches, and in this first case, compared with an exact solution. Secondly, a full 3D finite element model developed in ABAQUS, using interfaces between the concrete and the foundation soil, will be tested and compared with different methods to estimate the lateral group pile

displacement and maximum longitudinal stress in the pile. To validate the proposed analytical expressions, these will be later applied to examples tested by other authors in Chiou et al. [14], for group piles with both horizontal and bending moment.

4.1. Lateral deformations of a free-head pile

In this example, the existing models to estimate the equivalent pile length will be tested and compared with the exact value and the proposed formulation, admitting the analysis is performed in linear terms for both physical and geometrical behavior. First, it will be compared the values of the dimensionless lateral displacement and rotational ratio of each model with reference to the exact one Eq. (47).

This example comprises a singular free-head pile with a unique type of soil in a total length of 20 meters. It will be admitted an academic example with constant $k_s = 10000 \text{ kN/m}^2$ and linear variation for the elasticity $n_h = 10000 \text{ kN/m}^3$. For the pile, it was admitted a diameter of $D_p = 0.8 \text{ m}$ and concrete material with a Young's modulus of $E_c = 30 \text{ GPa}$.

$$\begin{aligned} r_{disp} &= \frac{\Delta_{model}}{\Delta_{exact}} \\ r_{rot} &= \frac{\theta_{model}}{\theta_{exact}} \end{aligned} \quad (47)$$

$$e = \frac{M_0}{H_0} \quad (48)$$

$$k_s; n_h \quad (49)$$

From Figures 10(a) and 11(a) it is possible to observe that the proposed method (PM) of this work presents an exact lateral displacement and head rotation when compared with the exact solution from the analytical solution provided by Winkler's differential Eq. (5). The LUEQ and LREQ both fail to catch either a correct lateral displacement or head rotation since its ratio is always above 1.5. Both the LG and US models present a more rigid solution for both deformations. In the case of LG it increases its flexibility with higher eccentricity, this is contrary to US model. In addition, both KUEQ and KDEQ models present higher flexibility than the exact model for both deformations. As expected, LC also presents the exact solution for the rotation of the head, but it is only an approximation for the lateral displacement. For this last deformation, it converges to the exact one if the eccentricity increases. This is due to being governed by bending moment when eccentricity increases.

For the soil with linear variation of elasticity with depth depicted in Figures 10(a) and 11(a), the conclusions are almost identical. The small difference is that for the rotation, LC no longer obtains the exact solution for the rotation in the pile head when the eccentricity increases.

The EEM method is not presented in either Figures 10 or 11, since it was already proven in the work of Chiou et al. [14], that this method also provided an exact equivalent pile flexibility, just like the PM method proposed in this work.

4.2. Maximum bending moment of a free-head pile

It will also be compared to the dimensionless maximum bending moment ratio of each model, again with reference to the exact one Eq. (50). To understand the differences between models, a parametric variation will be undertaken for eccentricity Eq. (48), for both soil reaction properties being either constant or linear Eq. (49), in which the different structural responses will be analyzed. For the maximum longitudinal bending moment, it was admitted for comparison that the four different possibilities, from four different models, are represented in Tables 3 and 4 for both type of soils.

$$r_{M_{\max}} = \frac{M_{\max}^{\text{exact}}}{M_{\max}^{\text{model}}} \quad (50)$$

In Figure 12(a), the maximum longitudinal bending moment in the pile of four several models are presented and compared with the exact solution again provided by the Winkler solution. Just by a quick observation, it seems that PM is able to achieve the same value as the exact solution. But, observing Figure 13(a), in which a zoom of the previous graph is illustrated, the PM for eccentricity between 1.0 and 20.0 presents a small upper bound below 1.05. In the case of Figure 12 b) for soils with linear variation of elasticity, the PM seems to also almost achieve the exact solution, in which some differences occur in the zoom of Figure 13(b), with a lower bond above 0.95. As expected, as demonstrated in section 2 and with expression (14), it was expected that the EEM model would presented an upper bound when estimating the maximum bending moment in the pile. The LG model also presents a consistent upper bound for both types of soils but converges for the exact

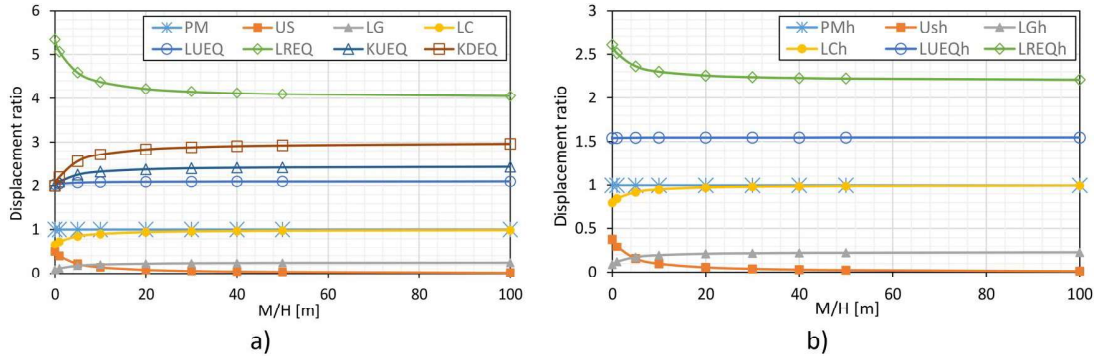


Figure 10. Displacement ratio for (a) soils with constant elasticity; (b) soils with variation linear of elasticity with depth.

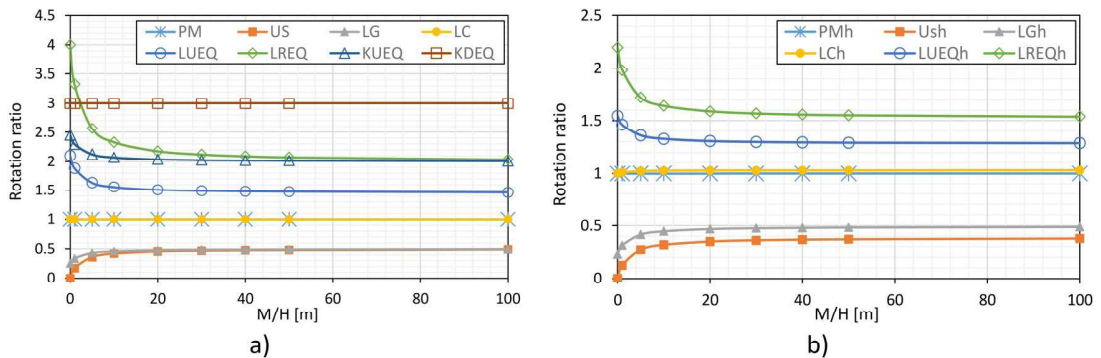


Figure 11. Rotation ratio for (a) soils with constant elasticity; (b) soils with linear variation of elasticity with depth.

Table 3. Maximum bending moment in the pile for soils with constant elasticity ($k_s = cte$).

Name	Expressions
Proposed Model (PM)	$M_{\max} = 0.64478H_0y_c \frac{H_0}{H_0 + 2M_0} + M_0$ (44)
Bending Moment of (LG) [26]	$M_{\max} = y_c H_0 + M_0$ (23)
Bending Moment of (AM) [20]	$M_{\max} = 0.3224 \frac{H_0}{\lambda} + 0.7M_0$ (17)
Bending Moment of (EEM) [14]	$M_{\max} = H_0 L_{eq} + M_0$ (30)

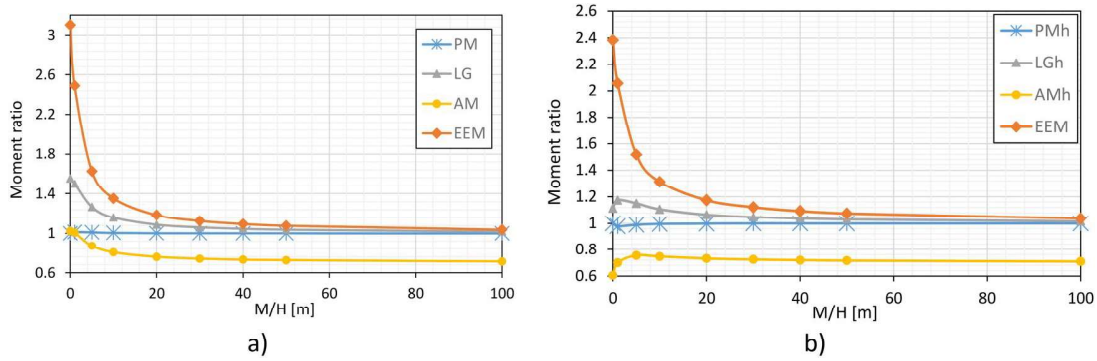
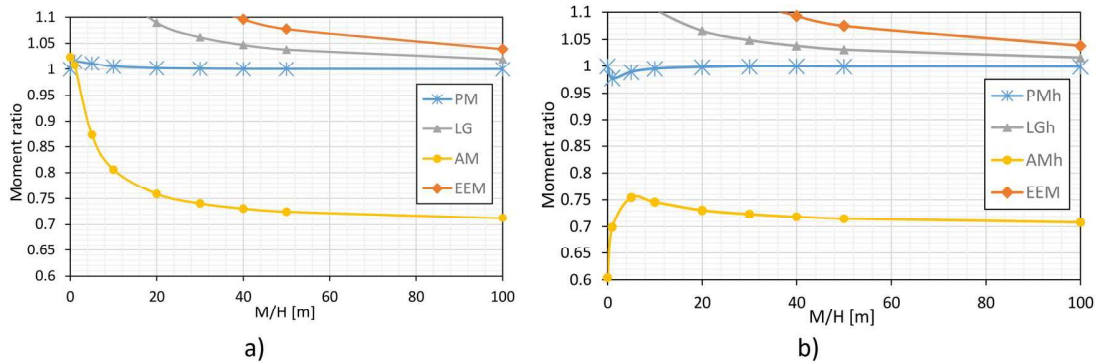
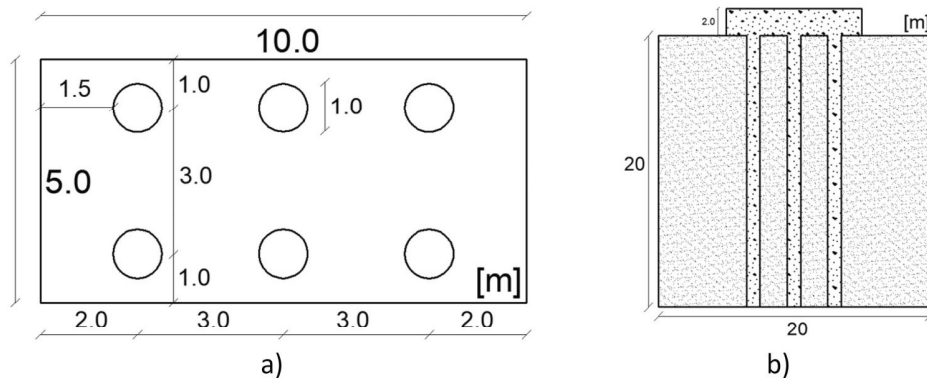
Table 4. Maximum bending moment in the soils with variation of elasticity with depth. ($k_s = n_h x$).

Name	Expressions
Proposed Model (PMh)	$M_{\max} = 0.833H_0y_{ch} \sqrt{\frac{H_0}{H_0 + \lambda_h M_0}} + M_0$ (45)
Bending Moment of (LGh) [26]	$M_{\max} = y_{ch} H_0 + M_0$ (24)
Bending Moment of (AMh) [20]	$M_{\max} = 0.3224 \frac{H_0}{\lambda_h} + 0.7M_0$ (17)
Bending Moment of (EEMh) [14]	$M_{\max} = H_0 L_{eqh} + M_0$ (30)

solution when increasing the eccentricity. The AM, since the bending moment in the head of the pile M_0 is limited to 0.7, it would be expected this approximation would always achieve a lower bond, and caution should be advised. In addition, this last model does not present a new analytical formulation for soils with linear variation of elasticity, therefore, for this case, it would be expected not to achieve the exact solution.

4.3. Maximum pile stress for a group pile

In this next example the proposed model will be compared with a full 3D solid finite element model using the commercial software ABAQUS Standard. It is admitted a simple academic geometry of a possible group pile made of reinforced concrete, with six supports Figure 14(a). For the purposes of

**Figure 12.** Moment ratio for (a) soils with constant elasticity; (b) soils with variation linear of elasticity with depth.**Figure 13.** Zoom of Figure 12 for the moment ratio for (a) soils with constant elasticity; (b) soils with variation linear of elasticity with depth.**Figure 14.** Pile geometry: (a) top view; (b) soil deepness and lateral cover.

simulation, it is admitted a soil cover of at least double of the maximum length of the pile to exclude any possible indirect interaction Figure 14(b). The total length of the pile was 20 meters to ensure a flexible behavior to be compared with the proposed formulation of this work.

For the proposed equivalent length model, it will use the structural representation of Figure 9(b). Using the properties presented in Table 5, it is possible to assemble a finite element model with SAP2000 Figure 15. The calculation of all properties are according to Eqs. (2), (37) and Eq. (44) in the previous sections of this work. For the concrete pile, a diameter $D_p = 1.0m$, resulting in a soil reaction parameter of $k_s = 4764kN/m^2$ according to expression (3) presented before. All finite elements are modeled as rigid offset, since for this proposed model the flexibility is governed only by the springs.

For the soil, two different parts were admitted. The first, a more refined one directly below the embedment, with similar mesh refinement Figure 16(b). The second part is the cover soil, which its only function is to provide lateral stiffness with a coarser mesh to reduce the degrees of freedom. These two parts are then connected using a tie interface with kinematic constraints. For the mesh, a 20 node

solid finite element named C3D20 with full $3 \times 3 \times 3$ Gauss integration was used for the concrete pile. However, for the soil foundation to decrease the degrees of freedom, the bilinear element with 8 nodes, C3D8 with full $2 \times 2 \times 2$ Gauss integration points, was adopted Figure 16(b). All materials used linear constitute relations with geometrically linear analysis to be directly compared with the proposed model. The properties of the materials are depicted in Table 6, these are academic values usually found in the civil engineering industry.

It was considered two different interfaces for the contact of concrete with the soil. The first interface allowed a cohesive and friction contact behavior between the concrete and the soil, admitting the possibility of slip-bond between the two materials. This interface was applied between the contact of the pile with the soil Figure 16(a). The second interface allowed a frictionless behavior between the concrete and the soil, admitting only a pressure behavior between both. This last interface was applied to simulate the contact between the embedment and the soil Figure 16(a). For the pressure behavior, for both interfaces, it was admitted hard contact behavior, with contact control using automatic stabilization. The properties of the interfaces are displayed in Table 7, these are academic values usually adopted in the civil engineering structural design.

To better represent the stress distribution in group piles using simplified methodologies, the analytical expressions proposed by Randolph [38] should be used for design purposes. In this particular case, this influence is small enough to be disregarded in the proposed equivalent pile length model. The model was fixed in the bottom for all materials and laterally confined perpendicular to the load Figure 17(a). To simulate the load that would produce horizontal force and bending moment, a lateral pressure of $100kN/m^2$ as applied Figure 17(b).

Table 5. Properties for the FEM with SAP2000.

$y_c[m]$	$K_A[kN/m]$	$K_\theta[kN/rad]$	$K_v[kN/m]$	$D[m]$
2.46	49259	298805	551000	1.0

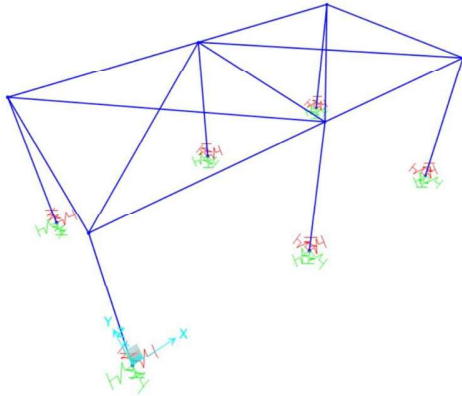


Figure 15. Geometry of the 6-pile embedment system FEM model with SAP2000.

Table 6. Material properties for the pile and soil.

Material	E_c	ν
Concrete	30GPa	0.2
Soil	10MPa	0.1

Table 7. Interface properties between the concrete and the soil.

Interface	σ_t	τ_c	k_t	k_c	μ	G_{III}
Soil-Pile	10kPa	50kPa	500MPa/mm	500MPa/mm	0.4	1000N/m
Soil-Embedment	—	—	—	—	0.4	—

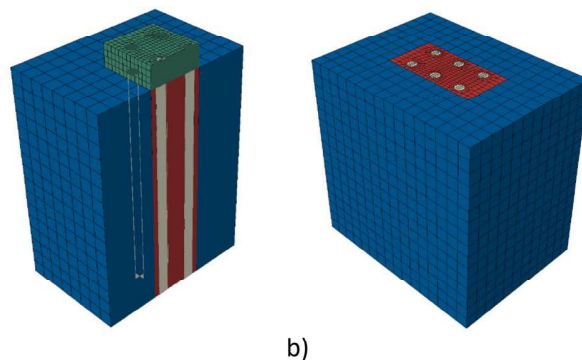
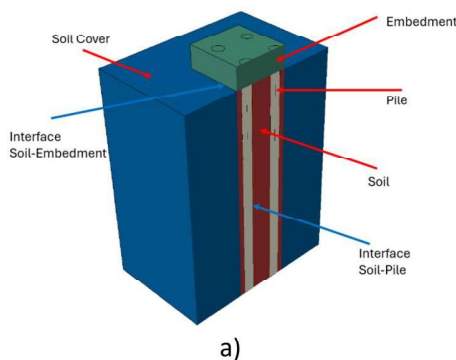


Figure 16. (a) Model geometry; (b) adopted finite element mesh.

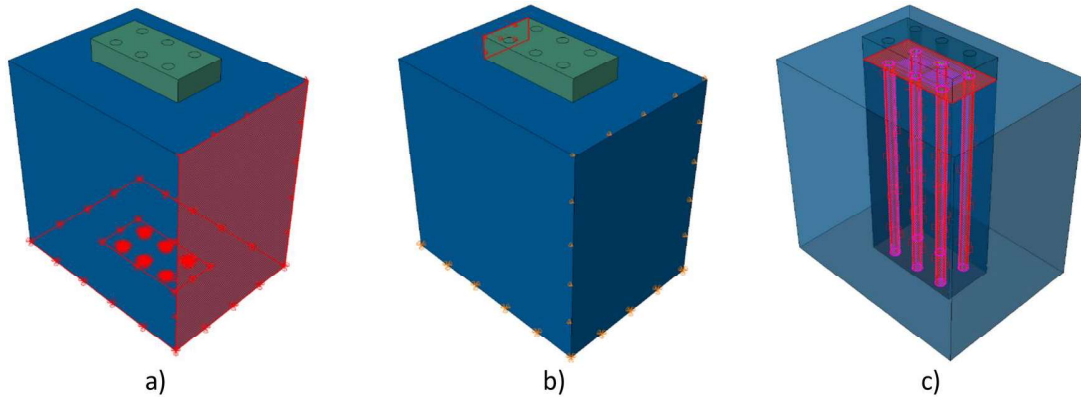


Figure 17. (a) Boundary conditions; (b) applied loads; (c) adopted interface surfaces.

Table 8. Maximum longitudinal stresses.

Model	Longitudinal stress
Equivalent Pile Length Model Proposed (PM)	5647kPa
3D Solid with perfect bond between pile and soil	5590kPa
3D Solid with interface between pile and soil	6127kPa

Two models were considered. The first admits a perfect bond if pressure exists between the concrete and the soil, which is close to the full linear elastic model of Winkler. The second considers a surface interface between concrete and soil with cohesive and friction non-linear properties Figure 17(c). The first model is supposed to present results similar to the proposed equivalent pile length model in this work. However, this may not be true for the second model due to the non-linear behavior.

Since the 3D solid model does not allow the calculation of bending moments in the pile, the maximum longitudinal stresses in the pile are directly compared with the proposed equivalent pile length model, according to expression Eq. (51). Since pile stresses are being represented by 3D solid continuum elements, some stress concentration is expected, which classical beam elements do not capture [39] (Table 8).

$$\sigma_{\max} = \frac{M_{\max}}{I} \left(\frac{D_p}{2} \right) \quad (51)$$

As expected, the proposed equivalent pile length model was able to capture the same maximum stress as the 3D solid model with a perfect bond between the concrete and the soil. This behavior is similar to the simulation of a pile with a Winkler foundation, therefore, it would be expected to present similar results. However, if it is admitted and interfaces between the concrete and the soil, with non-linear behavior, higher singular longitudinal stresses may arise. This is also expected, since this 3D model is more flexible and may cause some stress concentrations that are not captured by classical beam formulation. The global structural model is much simplified, and the number of elements is greatly decreased by employing the corresponding proposed equivalent pile length model.

4.4. Structural response for a group pile

In this last example, the benchmark case study executed by [14], in which the maximum bending moment in the pile

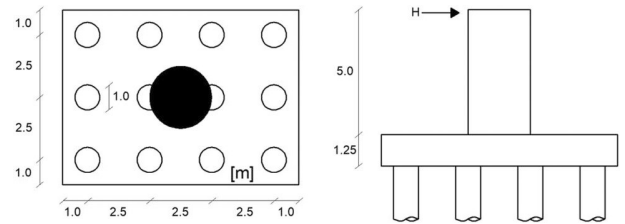


Figure 18. Geometry of pier and embedment.

cap of a group of piles is analyzed, using several methodologies. This example is mainly used to validate the proposed equivalent length model, with results provided by other authors.

It was admitted a concrete with $E_c = 27GPa$; $\nu = 0.2$, a soil with $E_s = 23MPa$; $\nu = 0.18$, an horizontal load of $H = 8000kN$ and a vertical load of $N = 15000kN$. The geometry of pier and embedment is represented in Figure 18, in which the pile possesses a total length of 30 meters. This example takes into account the effect of both horizontal load and bending moment applied in the embedment created by the pier height, resulting in a combination of stresses in the pile.

This case study was modeled initially using a beam in an elastic foundation with the Winkler Model. It was used a finite element model, in which the interaction of the soil with the pile was performed using discretized springs ($k_{springs} = k_s \times D$). A full detail of the finite element model geometry, including its applied load and boundary conditions, is fully described in Chiou et al. [14]. The results of this work will be directly compared with the proposed equivalent length model. It will also be used the results from other authors to estimate the equivalent pile length, some of which are presented in Table 1, to compare the maximum bending moment at the pile cap.

The structural representation of Figure 9(b) will be used for the proposed equivalent length model. With SAP2000 Figure 19, a finite element model can be put together using the properties shown in Table 9. All properties have been calculated in accordance with Eqs. (2), (37) and Eq. (44) in the earlier sections of this study. Since only the springs contribute to the flexibility in this proposed model, all of the finite elements are modeled as rigid offset. Using equivalent pile length models reduces the number of elements greatly, making the global structural model much simpler, even when the pile-embedment system uses simple 3D frame finite elements.

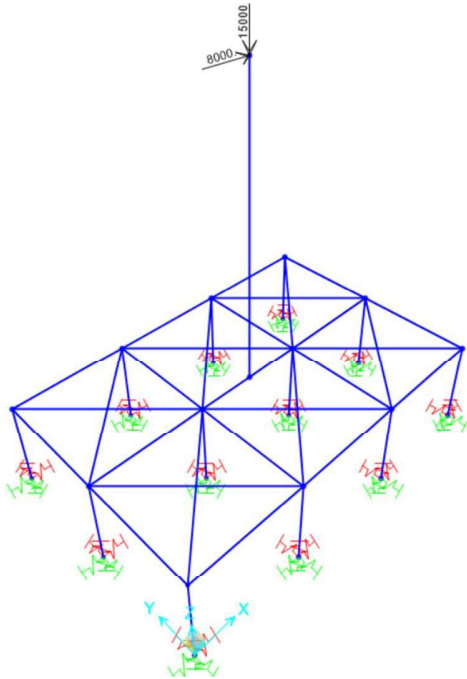


Figure 19. Geometry of the 12-pile embedment system FEM model with SAP2000.

Table 9. Properties for the FEM with SAP2000.

$y_c [m]$	$K_\Delta [kN/m]$	$K_\theta [kN/rad]$	$K_v [kN/m]$	$D [m]$
1.95	89606	340018	551000	1.0

Table 10. Comparison of the bending moments for all models [kNm].

Winkler Model	PM	EEM	US	LC	KUEQ
-898	-899.4	-898	569	-898	-1158

Table 11. Comparison of the horizontal displacement for all models [mm].

Winkler Model	PM	EEM	US	LC	KUEQ
9.7	9.74	9.7	7.4	4.8	9.8

The results from the original Winkler model, as indicated in Table 10, are related to the bending moments computed by the five equivalent models. It is evident that the results provided by the original Winkler model and the proposed equivalent length model will be the same. For the bending moment, both US and KUEQ Table 10, present coarse results when compared with the Winkler model, with the US changing the sign of the result. The results from other comparable models, however, are unsatisfactory in some cases, which is the case for the US and LC Table 11 for the horizontal displacement at the pile cap. Nonetheless, the suggested corresponding cantilever model can consistently produce precise outcomes, which is the case for the horizontal displacements in the pile cap Table 11. It is important to point out that the PM produces almost the same result as the Winkler Model and EEM. This is due to the accuracy of the models not being exactly the same since different finite element software was used in Chiou et al. [14].

It is evident from the aforementioned case study that the traditional equivalent models are unsatisfactory and that

altering the loading conditions will result in different errors. Applications in engineering are not appropriate for them.

5. Conclusions

Conventional equivalent models, such as the cantilever and uncoupled spring models, can be classified as approximation models in order to simulate the load-deflection behavior of a laterally loaded pile. These similar models cannot function equally well for various pile configurations and loading situations since their flexibility matrices do not exactly match those of the original Winkler model. Occasionally, this will result in extremely significant mistakes in the responses that are achieved.

The equivalent pile length model proposed in this work was able to estimate the lateral deformation of the pile and the maximum bending moment when compared with the exact solution provided by the differential equation of a pile embedded in an elastic medium. The proposed model was able to outstand existing simplified methodologies in terms of pile flexibility. Even compared with another exact equivalent pile length, it was able to almost capture the exact maximum bending moment in the pile. When comparing with a full 3D solid group pile model, with perfect bond between the concrete and the soil, the proposed equivalent pile length model was indeed successful. However, for the model with an interface between the concrete and the soil, since singular stresses may occur due to slip, it was not possible to provide a valid comparison.

In any case, the proposed equivalent pile length model can indeed be used for design purposes for structural engineers, with a full structure with pier, embedment and pile-soil system, with a high degree of confidence when compared with more robust numerical models.

5.1. Further developments

In this work the proposed formulation was not tested either for physically or geometrically non-linear analysis. For the case of geometrically non-linear analysis, there are already some works in which it is proven that rigid frames with nodal springs can indeed reproduce similar results to flexible frames when analyzing buckling [40,41], therefore, a good agreement of this formulation with non-geometrically non-linear analysis is expected, and this subject will be tested. Regarding the use of non-linear springs, to take into account the non-linear behavior of the sections is something already studied by the scientific community with good results [42]. In any case, both these topics will be pursued in the future with this new formulation.

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Disclosure statement

No potential conflict of interest was reported by the author(s).

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